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# EQUIVARIANT DEFINABLE MORSE FUNCTIONS ON DEFINABLE $C^{\infty}G$ MANIFOLDS

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ABSTRACT. Let $G$ be a compact affine definable $C^{\infty}$ group, $X$ a compact definable $C^{\infty}G$ manifold and $f$ an equivariant definable Morse function on $X$. We prove that if $f$ has no critical value in $[a, b]$, then $f^{-1}((-\infty, a])$ is definably $C^{\infty}G$ diffeomorphic to $f^{-1}((-\infty, b])$. Moreover we prove that if $r$ is a positive integer greater than 1, then the set of equivariant definable Morse functions on $X$ whose critical loci are finite unions of nondegenerate critical orbits is dense in the set of $G$ invariant $C^{\infty}$ functions on $X$ with respect to the $C^r$ Whitney topology.

## 1. INTRODUCTION

In this paper we consider an equivariant definable $C^{\infty}$ version of Morse theory. We refer the reader to the book by J. Milnor [16] for Morse theory on compact $C^{\infty}$ manifolds. Its equivariant versions are studied in G. Wasserman [21], K.H. Mayer [15], M. Datta and N. Pandey [1], and its definable $C^r$ versions are considered in T.L. Loi [14], Y. Peterzil and S. Starchenko [17] when $2 \leq r < \infty$.

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, e^x, \dots)$ be an exponential o-minimal expansion of $\mathbf{R}_{exp} = (\mathbb{R}, +, \cdot, <, e^x)$ admitting the $C^{\infty}$ cell decomposition. General references on o-minimal structures are [2], [3], see also [20]. It is known in [18] that there exist uncountably many o-minimal expansions of $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$.

Every definable $C^{\infty}$ manifold does not have boundary unless otherwise stated. Definable $C^rG$ manifolds are studied in [9], [7] when $0 \leq r \leq \omega$. Everything is considered in $\mathcal{M}$.

Let $G$ be a definable $C^{\infty}$ group, $X$ a definable $C^{\infty}G$ manifold and $f : X \to \mathbb{R}$ a $G$ invariant definable $C^{\infty}$ function on $X$. A closed definable $C^{\infty}G$ submanifold $Y$ of $X$ is called a *critical manifold* (resp. a *nondegenerate critical manifold*) of $f$ if each $p \in Y$ is a critical point (resp. a nondegenerate critical point) of $f$. We say that $f$ is an *equivariant definable Morse function* if the critical locus of $f$ is a finite union of nondegenerate critical manifolds of $f$ without interior.

**Theorem 1.1.** *Let $G$ be a compact affine definable $C^{\infty}$ group and $f$ an equivariant definable Morse function on a compact definable $C^{\infty}G$ manifold $X$. If $f$ has no critical value in $[a, b]$, then $f^a := f^{-1}((-\infty, a])$ is definably $C^{\infty}G$ diffeomorphic to $f^b := f^{-1}((-\infty, b])$.*

Theorem 1.1 is an equivariant definable version of Theorem 4.3 [21] and a definable $C^{\infty}$ version of 1.1 [6].

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In the non-equivariant definable case, T.L. Loi [14] proves the density of definable Morse functions.

Let $r$ be a positive integer greater than 1, $Def^r(\mathbb{R}^n)$ denote the set of definable $C^r$ functions on $\mathbb{R}^n$. For each $f \in Def^r(\mathbb{R}^n)$ and for each positive definable continuous function $\epsilon : \mathbb{R}^n \to \mathbb{R}$, the $\epsilon$-neighborhood $N(f;\epsilon)$ of $f$ in $Def^r(\mathbb{R}^n)$ is defined by $\{h \in Def^r(\mathbb{R}^n) | |\partial^\alpha(h-f)| < \epsilon, \forall \alpha \in (\mathbb{N} \cup \{0\})^n, |\alpha| \leq r\}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n, |\alpha| = \alpha_1 + \cdots + \alpha_n, \partial^\alpha F = \frac{\partial^{|\alpha|}F}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$. We call the topology defined by these $\epsilon$-neighborhoods the *definable $C^r$ topology*.

**Theorem 1.2** ([14]). *Let $r$ be a positive integer greater than 1 and $X$ a definable $C^r$ submanifold of $\mathbb{R}^n$. Then the set of definable $C^r$ functions on $\mathbb{R}^n$ which are Morse functions on $X$ and have distinct critical values are open and dense in $Def^r(\mathbb{R}^n)$ with respect to the definable $C^r$ topology.*

Remark that the definable $C^r$ topology and the $C^r$ Whitney topology do not coincide in general. If $X$ is compact, then these topologies of the set $Def^r(X)$ of definable $C^r$ functions on $X$ are the same (P156 [20]).

A nondegenerate critical manifold of an equivariant Morse function on a definable $C^\infty G$ manifold is called a *nondegenerate critical orbit* if it is an orbit. The following is the density of equivariant definable Morse functions.

**Theorem 1.3.** *Let $G$ be a compact affine definable $C^\infty$ group, $X$ a compact definable $C^\infty G$ manifold and $r$ a positive integer greater than 1. Then the set $Def_{equi-Morse,o}(X)$ of equivariant definable Morse functions on $X$ whose critical loci are finite unions of nondegenerate critical orbits is dense in the set $C^\infty_{inv}(X)$ of $G$ invariant $C^\infty$ functions on $X$ with respect to the $C^r$ Whitney topology. Moreover $Def_{equi-Morse,o}(X)$ is open and dense in the set $Def^\infty_{inv}(X)$ of $G$ invariant definable $C^\infty$ functions with respect to the definable $C^r$ topology.*

The following is a definable $C^\infty$ version of a well-known topological result (e.g. 6.2.4 [5]).

**Theorem 1.4.** *Let $X$ be an $n$-dimensional compact definable $C^\infty$ manifold admitting a definable Morse function $f : X \to \mathbb{R}$ with only two critical points.*

(1) *([6]) $X$ is definably homeomorphic to the $n$-dimensional unit sphere $S^n$.*
(2) *If $n \leq 6$, then $X$ is definably $C^\infty$ diffeomorphic to $S^n$.*

## 2. Proof of Theorem 1.1

A *definable $C^\infty$ manifold* is a $C^\infty$ manifold with a finite system of charts whose transition functions are definable, and definable $C^\infty$ maps, definable $C^\infty$ diffeomorphisms and definable $C^\infty$ imbeddings are defined similarly ([9], [7]). A definable $C^\infty$ manifold is *affine* if it is definably $C^\infty$ imbeddable into some $\mathbb{R}^n$. If $\mathcal{M} = \mathcal{R}$, a definable $C^\omega$ manifold (resp. an affine definable $C^\omega$ manifold) is called a *Nash manifold* (resp. an *affine Nash manifold*). By [8], every definable $C^r$ manifold is affine when $r$ is a non-negative integer. The definable $C^\omega$ case is complicated. Even if $\mathcal{M} = \mathcal{R}$, it is known that for every compact or compactifiable $C^\omega$ manifold of positive dimension admits a continuum number of distinct nonaffine Nash manifold structures [19], and its equivariant version is proved in [10].

A group $G$ is a *definable* $C^\infty$ *group* if $G$ is a definable $C^\infty$ manifold such that the group operations $G \times G \to G$ and $G \to G$ are definable $C^\infty$ maps. By definition, every definable $C^\infty$ group is a Lie group. Let $G$ be a definable $C^\infty$ group. A *definable* $C^\infty G$ *manifold* is a pair $(X, \phi)$ consisting of a definable $C^\infty$ manifold $X$ and a group action $\phi : G \times X \to X$ such that $\phi$ is a definable $C^\infty$ map. For simplicity, we write $X$ instead of $(X, \phi)$.

Let $G$ be a definable $C^\infty$ group. A *representation map* of $G$ means a group homomorphism from $G$ to some $O_n(\mathbb{R})$ which is a definable $C^\infty$ map and the *representation* of this representation map is $\mathbb{R}^n$ with the orthogonal action induced by the representation map. In this paper, we always assume that every representation is orthogonal. A *definable* $C^\infty G$ *submanifold* of a representation $\Omega$ of $G$ is a $G$ invariant definable $C^\infty$ submanifold of $\Omega$. We say that a definable $C^\infty G$ manifold is *affine* if it is definably $C^\infty G$ diffeomorphic to a definable $C^\infty G$ submanifold of some representation of $G$.

In our assumption, every compact definable $C^\infty G$ manifold is affine.

**Theorem 2.1** ([9]). *Let $G$ be a compact affine definable $C^\infty$ group. Then every compact definable $C^\infty G$ manifold is affine.*

Remark that if $\mathcal{M}$ is polynomially bounded, then Theorem 2.1 is not always true [10].

**Theorem 2.2.** *Let $G$ be a compact affine definable $C^\infty$ group. Let $X$ and $Y$ be compact definable $C^\infty G$ manifolds possibly with boundary. If either $\partial X = \partial Y = \emptyset$ or $X, Y$ are affine, then $X$ and $Y$ are definably $C^\infty G$ diffeomorphic if and only if they are $C^1 G$ diffeomorphic.*

To prove Theorem 2.2, we prepare several results.

**Theorem 2.3** (2.24 [7]). *Let $G$ be a compact definable $C^\infty$ group.*
*(1) Every definable $C^\infty G$ submanifold $X$ possibly with boundary of a representation $\Omega$ of $G$ has a definable $C^\infty G$ tubular neighborhood $(U, p)$ of $X$ in $\Omega$.*
*(2) Any compact affine definable $C^\infty G$ manifold $X$ with boundary $\partial X$ admits a definable $C^\infty G$ collar, namely there exists a definable $C^\infty G$ imbedding $\phi : \partial X \times [0,1) \to X$ such that $\phi(\partial X \times [0,1))$ is a $G$ invariant definable open neighborhood of $\partial X$ in $X$ and $\phi(x, 0) = x$ for all $x \in \partial X$, where the action on the closed unit interval $[0,1]$ is trivial.*

Let $G$ be a compact definable $C^\infty$ group. Let $f$ be a map from a $C^\infty G$ manifold $X$ to a representation $\Omega$ of $G$. Denote the Haar measure of $G$ by $dg$ and let $C^\infty(X, \Omega)$ denote the set of $C^\infty$ maps from $X$ to $\Omega$. Define

$$A : C^\infty(X, \Omega) \to C^\infty(X, \Omega), A(f)(x) = \int_G g^{-1} f(gx) dg.$$

We call $A$ the *averaging function*. In particular, if $G = \{g_1, \dots, g_n\}$, then $A(f)(x) = \frac{1}{n} \sum_{i=1}^n g_i^{-1} f(g_i x)$.

Observations similar to 2.6 [12], 4.3 [7] and 2.35 [13] show the following proposition.

**Proposition 2.4** ([12], [7], [13]). *Let $G$ be a compact definable $C^\infty$ group.*
*(1) $A(f)$ is equivariant, and $A(f) = f$ if $f$ is equivariant.*

*(2) If $0 \leq r \leq \infty$ and $f \in C^r(X, \Omega)$, then $A(f) \in C^r(X, \Omega)$.*
*(3) If $f$ is a polynomial map, then so is $A(f)$.*
*(4) If $0 \leq r < \infty$ and $X$ is compact, then $A : C^r(X, \Omega) \to C^r(X, \Omega)$ is continuous in the $C^r$ Whitney topology.*
*(5) If $G$ is a finite group, $X$ is a definable $C^\infty G$ manifold and $f$ is a definable $C^\infty$ map, then $A(f)$ is a definable $C^\infty G$ map.*

**Theorem 2.5** (P 38 [5]). *(1) Let $X, Y$ be $C^1$ manifolds. Then the set of $C^1$ diffeomorphisms from $X$ onto $Y$ is open in the set $C^1(X, Y)$ of $C^1$ maps from $X$ to $Y$ with respect to the $C^1$ Whitney topology.*
*(2) Let $X, Y$ be $C^1$ manifolds with boundary $\partial X, \partial Y$, respectively. Then the set of $C^1$ diffeomorphisms from $X$ onto $Y$ is open in $\{f \in C^1(X, Y) | f(\partial X) \subset f(\partial Y)\}$ with respect to the $C^1$ Whitney topology.*

**Theorem 2.6** (1.2 [4]). *Let $A, B$ be definable disjoint closed subsets of $\mathbb{R}^n$. Then there exists a definable $C^\infty$ function $\phi : X \to \mathbb{R}$ such that $\phi|A = 1$ and $\phi|B = 0$.*

The following is an equivariant version of Theorem 2.6.

**Theorem 2.7** ([11]). *Let $G$ be a compact definable $C^\infty$ group and $X$ a compact definable $C^\infty G$ manifold. Suppose that $A, B$ are $G$ invariant definable disjoint closed subsets of $X$. Then there exists a $G$ invariant definable $C^\infty$ function $f : X \to \mathbb{R}$ such that $f|A = 1$ and $f|B = 0$.*

Remark that if $\mathcal{M}$ is polynomially bounded, then Theorem 2.6 and 2.7 are not always true.

*Proof of Theorem* 2.2. Assume first that $\partial X = \partial Y = \emptyset$. By Theorem 2.1, we may assume that $X, Y$ are definable $C^\infty G$ submanifolds of a representation $\Omega$ of $G$. Using Theorem 2.3, we have a definable $C^\infty G$ tubular neighborhood $(U, p)$ of $Y$ in $\Omega$.

Let $f : X \to Y$ be a $C^1 G$ diffeomorphism and $i : Y \to \Omega$ the inclusion. Applying the polynomial approximation theorem, we have a polynomial map $f' : X \to \Omega$ as a $C^1$ approximation of $i \circ f$. Applying the Haar measure and Proposition 2.4, there exists a polynomial $G$ map $f'' : X \to \Omega$ approximating $i \circ f$. If this approximation is sufficiently close, then $i \circ f(X) \subset U$. By Proposition 2.4, $F := p \circ f'' : X \to Y$ is a definable $C^\infty G$ map which is a $C^1$ approximation of $f$. Hence using Theorem 2.5 and the inverse function theorem, $F : X \to Y$ is a definable $C^\infty G$ diffeomorphism.

We now prove the second case. By Theorem 2.3, we have definable $C^\infty G$ collar neighborhoods $\phi_X : \partial X \times [0, 1) \to X, \phi_Y : \partial Y \times [0, 1) \to Y$ of $\partial X, \partial Y$ in $X, Y$, respectively.

By the first argument, we have a definable $C^\infty G$ diffeomorphism $F_{\partial X} : \partial X \to \partial Y$ as a $C^1$ approximation of $f|\partial X$. Using these definable $C^\infty G$ collar neighborhoods, we have a definable $C^\infty G$ diffeomorphism $L_1 : \phi_X(\partial X \times [0, 1)) \to \phi_Y(\partial Y \times [0, 1))$ as a $C^1$ approximation of $f|\phi_X(\partial X \times [0, 1))$. Since $X - \phi(\partial X \times [0, \frac{3}{4}))$ is a compact definable $C^\infty G$ manifold with boundary and by the first argument, there exists a definable $C^\infty G$ map $L_2 : X - \phi(\partial X \times [0, \frac{3}{4})) \to Y$ as a $C^1$ approximation of $f|(X - \phi(\partial X \times [0, \frac{3}{4})))$. By Theorem 2.7, we have a $G$ invariant definable $C^\infty$ function $k : X \to \mathbb{R}$ such that

$k|\phi(\partial X \times [0, \frac{1}{3}]) = 1$ and $k|(X - \phi(\partial X \times [0, \frac{1}{2}))) = 0$. Thus the map $H : X \to Y$ defined by

$$H(x) = \begin{cases} p(k(t)L_1(x) + (1 - k(t))L_2(x)), & x \in \phi_X(\partial X \times [0,1)) \\ L_2(x), & x \in X - \phi_X(\partial X \times [0,1)) \end{cases}$$

is a definable $C^\infty G$ map such that $H(\partial X) = \partial Y$ and $H$ is a $C^1$ approximation of $f$. Therefore $H$ is the required definable $C^\infty G$ diffeomorphism. □

*Proof of Theorem* 1.1. By Theorem 4.3 [21], $f^a = f^{-1}((-\infty, a])$ is $C^\infty G$ diffeomorphic to $f^b = f^{-1}((-\infty, b])$. Since $X$ is compact and by Theorem 2.1, these two manifolds are compact affine definable $C^\infty G$ manifolds with boundary. Thus Theorem 2.2 proves Theorem 1.1. □

Remark that the method of the proof Theorem 4.3 [21] is the integration of a $G$ invariant $C^\infty$ vector field. This method does not work in the definable setting because the integration of a $G$ invariant definable $C^\infty$ vector field is not always definable.

## 3. Proof of Theorem 1.3 and 1.4

**Theorem 3.1** ([21]). *Let $G$ be a compact Lie group and $X$ a compact $C^\infty G$ manifold. Then the set $C^\infty_{equi-Morse,o}(X)$ of equivariant Morse functions on $X$ whose critical loci are finite unions of nondegenerate critical orbits is open and dense in the set $C^\infty_{inv}(X)$ of $G$ invariant $C^\infty$ functions on $X$ with respect to the $C^\infty$ Whitney topology.*

*Proof of Theorem* 1.3. Let $f \in C^\infty_{inv}(X)$ and $\mathcal{N} \subset C^\infty_{inv}(X)$ an open neighborhood of $f$ in $C^\infty_{inv}(X)$. By Theorem 3.1, there exists an open subset $\mathcal{N}' \subset \mathcal{N}$ such that each $h \in \mathcal{N}'$ is an equivariant Morse function whose critical locus is a finite union of nondegenerate critical orbits. Let $C^\infty(X)$ denote the set of $C^\infty$ functions on $X$. Since $A : C^\infty(X) \to C^\infty(X)$ is continuous and $A(C^\infty(X)) = C^\infty_{inv}(X)$, $A : C^\infty(X) \to C^\infty_{inv}(X)$ is continuous. Fix $h \in \mathcal{N}'$. Since $A(h) = h$, $A^{-1}(\mathcal{N}')$ is an open neighborhood of $h$ in $C^\infty(X)$. Applying the polynomial approximation theorem, we have a polynomial function $h'$ lies in $A^{-1}(\mathcal{N}')$. Applying the averaging function, we have a $G$ invariant polynomial function $F := A(h')$ lies in $\mathcal{N}'$. Since $F$ is a $G$ invariant polynomial function, it is a $G$ invariant definable $C^\infty$ function. Thus $F$ is an equivariant definable Morse function lies in $\mathcal{N}$.

We now prove the second part. By the first part, $Def_{equi-Morse,o}(X)$ is dense in $C^\infty_{inv}(X)$. Thus it is dense in $Def^\infty_{inv}(X)$.

Let $h \in Def_{equi-Morse,o}(X)$. By Theorem 3.1, there exists an open neighborhood $\mathcal{V}$ of $h$ in $C^\infty_{inv}(X)$ such that each $h \in \mathcal{V}$ is an equivariant Morse function whose critical locus is a finite union of nondegenerate critical orbits. Thus $\mathcal{V} \cap Def_{inv}(X)$ is the required open neighborhood of $h$ in $Def_{inv}(X)$. □

*Proof of Theorem* 1.4. Using classical results, if $n \leq 6$, then $X$ is $C^\infty$ diffeomorphic to $S^n$. Thus since $X$ is compact and by Theorem 2.2, $X$ is definably $C^\infty$ diffeomorphic to $S^n$. □

## References

[1] M. Datta and N. Pandey, *Morse theory on G-manifolds*, Topology Appl. **123** (2002), 351–361.
[2] L. van den Dries, *Tame topology and o-minimal structures*, Lecture notes series **248** London Math. Soc. Cambridge Univ. Press (1998).
[3] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497–540.
[4] A. Fischer, *Smooth functions in o-minimal structures*, Adv. Math. **218** (2008), 496–514.
[5] M.W. Hirsch, *Differential topology*, Graduate Texts in Mathematics **33**, Springer-Verlag (1976).
[6] T. Kawakami, *Equivariant definable Morse functions on definable $C^rG$ manifolds*, Far East J. Math. Sci. (FJMS) **28** (2008), 175–188.
[7] T. Kawakami, *Equivariant differential topology in an o-minimal expansion of the field of real numbers*, Topology Appl. **123** (2002), 323–349.
[8] T. Kawakami, *Every definable $C^r$ manifold is affine*, Bull. Korean Math. Soc. **42** (2005), 165–167.
[9] T. Kawakami, *Imbeddings of manifolds defined on an o-minimal structures on* $(\mathbb{R}, +, \cdot, <)$, Bull. Korean Math. Soc. **36** (1999), 183–201.
[10] T. Kawakami, *Nash G manifold structures of compact or compactifiable $C^\infty G$ manifolds*, J. Math. Soc. Japan **48** (1996) 321–331.
[11] T. Kawakami, *Relative properties of definable $C^\infty$ manifolds with finite abelian group actions in an o-minimal expansion of $R_{\exp}$*, Bull. Fac. Ed. Wakayama Univ. Natur. Sci. **59** (2009), 21–27.
[12] T. Kawakami, *Simultaneous Nash structures of a compactifiable $C^\infty G$ manifold and its $C^\infty G$ submanifolds*, Bull. Fac. Edu. Wakayama Univ. Natur. Sci. **49** (1999), 1-20.
[13] K. Kawakubo, *The theory of transformation groups*, Oxford Univ. Press, 1991.
[14] T.L. Loi, *Density of Morse functions on sets definable in o-minimal structures*, Ann. Polon. Math. **89** (2006), 289–299.
[15] K.H. Mayer, *G-invariante Morse-funktionen*, Manuscripta Math. **63** (1989), 99–114.
[16] J. Milnor, *Morse theory*, Princeton Univ. Press (1963).
[17] Y. Peterzil and S. Starchenko, *Computing o-minimal topological invariants using differential topology*, Trans. Amer. Math. Soc. **359** (2007), 1375–1401.
[18] J.P. Rolin, P. Speissegger and A.J. Wilkie, *Quasianalytic Denjoy-Carleman classes and o-minimality*, J. Amer. Math. Soc. **16** (2003), 751–777.
[19] M. Shiota, *Abstract Nash manifolds*, Proc. Amer. Math. Soc. **96** (1986), 155–162.
[20] M. Shiota, *Geometry of subanalytic and semialgebraic sets*, Progress in Math. **150** (1997), Birkhäuser.
[21] G. Wasserman, *Equivariant differential topology*, Topology **8** (1969), 127–150.

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